Paper Code AS-2229

M.A/M.Sc. IIIrd Sem. Exams 2013

Mathematics

Paper IIIrd Optional

Fuzzy Sets & their applications.

(Suggested Solution).

1 (a) If A is a subset of ~~the~~ universal set X then

$A \cap \bar{A} = \phi$ is called the law of contradiction

& $A \cup \bar{A} = X$ is called the law of excluded middle.

(b) If A and B are two fuzzy set over a set X

then $(A \cup B)(x) = \max(A(x), B(x))$

$= \max(B(x), A(x))$

$= (B \cup A)(x) \quad \forall\, x \in X$

$\therefore \quad A \cup B = B \cup A$, the completes the proof.

(c) Let A, B and C be fuzzy sets over a set X, then we have: (using standard definitions)

$(A \cup (B \cap C))(x) = \max(A(x), \min(B(x), C(x)))$

$= \min(\max(A(x), B(x)), \max(A(x), C(x)))$

$= \min((A \cup B)(x), (A \cup C)(x))$

$= ((A \cup B) \cap (A \cup C))(x) \quad \forall\, x \in X$

$\therefore \quad A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

Illy students may derive the second distributive law.

(d) Let A and B be two fuzzy sets over X. Then

$$x \in {}^{\alpha}(A \cup B) \iff (A \cup B)(x) \geq \alpha$$
$$\iff \max(A(x), B(x)) \geq \alpha$$
$$\iff A(x) \geq \alpha \text{ or } B(x) \geq \alpha$$
$$\iff x \in {}^{\alpha}A \text{ or } x \in {}^{\alpha}B$$
$$\iff x \in {}^{\alpha}A \cup {}^{\alpha}B$$
$$\therefore \; {}^{\alpha}(A \cup B) = {}^{\alpha}A \cup {}^{\alpha}B$$

(e) Since ${}^{\alpha}A \subseteq {}^{\alpha}B$ $\alpha \in [0,1]$ it follows that $A(x) \geq \alpha \Rightarrow B(x) \geq \alpha$. From this students may choose his own method to conclude $A \subseteq B$.

(f) Here student is supposed to show that if $A_1 \subseteq A_2$ then $f(A_1) \subseteq f(A_2)$ using extension principle (and his own method) where A_1 and A_2 are any fuzzy sets over a universal set X

(g) Here student is supposed to prove that (under standard notations) $d_{e_c} = e_c$. Suggested proof is: If $a = e_c$ then by definition of equilibrium point $c(a) = a$ and so $a - c(a) = 0$, may conclude that $c(d_a) = d_a$. further student

$$\therefore \; c(d_a) - d_a = a - c(a)$$

Thus definition of dual point inforces that

$d_a = e_c$ i.e. $d_{e_c} = e_c$.

(h) An increasing generator is a continuous function from $[0, 1]$ to $\mathbb{R}$ (the set of real numbers) which is strictly increasing and such that $g(0) = 0$

(i) Since α- cuts of any fuzzy number are required ~~to be closed sets intervals $\forall \alpha \in$~~ to be closed intervals $\forall \in [0, 1]$ it follows by definition of convex fuzzy sets that every fuzzy number must be a convex fuzzy set.

(j) There are many possible fuzzy ordering relations. However the fuzzified version of partial ordering relation may be chosen as most applicable fuzzy ordering relation. Thus a fuzzy relation which is reflexive antisymmetric and transitive under some form of fuzzy transitivity may be termed as a fuzzy ordering relation.

(Marks Dist. $2 \times 10 = 20$)

2 (a) Here the student is supposed to give a detailed classification of fuzzy sets in his own language. However the answer must cover at least

(i) Ordinary Fuzzy sets

(ii) Interval-valued fuzzy sets

(iii) Fuzzy sets of Type-1 and of Type-2 and of higher Type.

(iv) Level 1 and level 2 fuzzy sets

He/She may combine the types and levels as per his/her choice.

Total M.M. = 5

2(b) Example of the formula (suggested) in

$$S(A,B) = \frac{|A \cap B|}{|A|}$$

Formula

$$|C| = \sum_{x \in \{1,2,\dots,10\}} \frac{x}{x+1}$$ may be ~~so~~ used to

calculate $|C| = \frac{1}{2} + \frac{2}{3} + \dots + \frac{10}{11} = 7.975$

& llly $|D| = (1-\frac{1}{10}) + (1-\frac{2}{10}) + (1-\frac{3}{10}) + \dots + (1-\frac{10}{10})$

$= 10 - 5.5 = 4.5$

(Marks Dist $1+2+2 = 5$)

3(a) Statement: For every $A \in \mathcal{F}(X)$ we have

$$A = \bigcup_{\alpha \in [0,1]} {}_{\alpha}A$$ where

${}_{\alpha}A = \alpha \cdot {}^{\alpha}A(x)$ and $\cup$ is the ~~standd~~ standard fuzzy union.

Proof: For any $x \in X$ and $A(x) = a$ say we have

$$\left(\bigcup {}_{\alpha}A\right)(x) = \sup_{\alpha \in [0,1]} {}_{\alpha}A(x)$$

$$= \max\left[\sup_{\alpha \in [0,a]} {}_{\alpha}A(x), \sup_{\alpha \in (a,1]} {}_{\alpha}A(x)\right].$$

Here for each ~~$\alpha \in$~~ $\alpha \in (a,1]$ $A(x) = a < \alpha$ and so ${}_{\alpha}A(x) = 0$, while for each $\alpha \in [0,a]$ we have $A(x) = a \geq \alpha$ so that ${}_{\alpha}A(x) = \alpha$. Hence

$$\left(\bigcup_{\alpha \in [0,1]} {}_{\alpha}A\right)(x) = \sup_{\alpha \in [0,a]} \alpha = a = A(x) \quad \forall x \in X$$

Therefore the result is seen to be true.

Marks Dist $1+4 = 5$

3 (b) Student is supposed to coves at least :

(i) Preservation of convexity

(ii) Preservation of finite union, finite intersection & discuss the infinite case in short

(iii) Establishment of decomposition theorems with the help of α- cuts.

(iv) Extension principle is strong cutworthy but not cutworthy

No proof of theorems are needed. Only relevent statements are sufficient. To give difference student may give definitions and also may observe that ${}^{\alpha+}A \subseteq {}^{\alpha}A$. Further the case $\alpha = 0$ and $\alpha = 1$ may be covered. M.M. 5 = 4+1 for difference.

4 (a) Student may give an example of his/her choice to ~~establishment~~ explain the situation in his/her words. M. M. 5

(b)

(i) $\forall\ y \in Y$ we have

$$
\begin{aligned}
y \in {}^{\alpha+}(f(A)) &\iff [f(A)](y) > \alpha \\
&\iff \sup_{x \in X \mid f(x) = y} A(x) > \alpha \\
&\iff \exists\ x_0 \in X \text{ such that } f(x_0) = y \text{ and } A(x_0) > \alpha \\
&\iff \exists\ x_0 \in X \text{ such that } f(x_0) = y \text{ and } x_0 \in {}^{\alpha+}A \\
&\iff y \in f({}^{\alpha+}A)
\end{aligned}
$$

Therefor we have ${}^{\alpha+}(f(A)) = f({}^{\alpha+}A)$

(ii) If $y \in f({}^{\alpha}A)$, then $\exists$ $x_0 \in {}^{\alpha}A$ such that $y = f(x_0)$.

Hence $[f(A)](y) = \sup_{x \in X | f(x) = y} A(x) \geq A(x_0) \geq \alpha$

and so $y \in {}^{\alpha}(f(A))$. Thus $f({}^{\alpha}A) \subseteq {}^{\alpha}[f(A)]$.

(Marks Dist. 3 + 2 = 5)

5 (a) The equilibrium of a complement c is that degree of membership in a fuzzy set A which equals the degree of membership in the complement cA. Mathematically if $A(x) = a$ then a is said to be an equilibrium of a complement c if $c(a) = a$. Now the student is supposed to give existance theorem and uniqueness theorem in his/her own ~~wo~~ words and way. No detailed proofs. are required

Marks Dist. 1 + 2 + 2 = 5

(b) Yager class of t-conorms is defined as:

$$u_w(a,b) = \min\left(1, (a^w + b^w)^{\frac{1}{w}}\right) \quad w > 0$$

Student is supposed to show that for different choices of w, $u_w(a,b)$ approches $\max(a,b)$ and $u_{max}(a,b)$ $\forall$ $a, b \in [0,1]$. He/She may also give different cases

Marks Dist For Lower bound 4 Marks
For Upper bound 1 Mark
Total 5 Marks.

6 (a) ~~B~~ Statement : Let c be a function from $[0,1]$ to $[0,1]$. c is a fuzzy (involutive) complement if and only if $\exists$ a continuous function g from $[0,1]$ to $\mathbb{R}$ such that $g(0) = 0$, g is strictly increasing and $c(a) = g^{-1}(g(1) - g(a))$ for all $a \in [0,1]$.

Proof : Suppose first that such a function g is given. The student may first define the psudo-inverse of g. Then he/she may verify the skeleton axioms to define a fuzzy complement using the given properties of g. He/She may skip over all derivations using known results.

For the converse, first student may observe that c has a unique equilibrium say e_c. Using e_c he/she may define / obtain a continuous, strictly increasing bijective map say $h : [0, e_c] \to [0, b]$ with $h(0) = 0$ $h(e_c) = b$ where b is any fixed positive real numbers. Using h student may define a function g in such a way that all the required conditions are satisfied by g and its psudo-inverse.

Marks Dist. $1 + 2 + 2 = 5$ Marks

6 (b) The generalized means are defined by the formula

$$h_\alpha(a_1, a_2, \ldots, a_n) = \left(\frac{a_1^\alpha + a_2^\alpha + \cdots + a_n^\alpha}{n}\right)^{1/\alpha}$$

Now student may show that

(1) h_α ~~satisfied~~ satisfies the five axioms to define aggregation operation

(ii) For different choices of α, h_α becomes geometric mean, harmonic mean and arithmetic mean

Marks Dist. $3+2 = 5$

7. Using the standard definition of addition and multiplication of intervals student may demonstrate that for both operations commutative laws hold. He/She demonstrate in short using the commutative property for addition and multiplication of real numbers. To check subdistributive law he may /she may show :

$$\begin{aligned} A.(B+C) &= \{a.(b+c) \mid a \in A,\ b \in B,\ c \in C\} \\ &= \{a.b + a.c \mid a \in A,\ b \in B,\ c \in C\} \\ &\subseteq \{a.b + a'.c \mid a \in A,\ a' \in A,\ b \in B,\ c \in C\} \\ &= A.B + A.C \end{aligned}$$

No detailed calculation needed

Marks Dist. $3 + 3 + 4 = 10$

8. The student may first state the following result:

Let $P(X, Y)$, $Q(Y, Z)$, $R(X, Z)$ be fuzzy relations then the following equations are equivalent:

(i) $P \overset{i}{\circ} Q \subseteq R$

(ii) $Q \subseteq P^{-1} \overset{w_i}{\circ} R$

(iii) $P \subseteq (Q \overset{w_i}{\circ} R^{-1})^{-1}$

(No proof needed)

He/She now may write in his/her way the proof of all the four parts asked in the question.

Marks Distribution $2.5 \times 4 = 10$ Marks.

Ranadive